Generalised MGF of the $\kappa - \mu$ extreme distribution and its applications to performance analysis

J. Gong, H. Lee, M. Park, J.W. Choi and J. Kang

The authors formulate the closed-form expressions of the generalised moment generating function (G-MGF) for the $\kappa - \mu$ extreme distribution, which enables one to calculate important metrics of wireless communications systems. The derived formula is utilised to evaluate the performance of communication systems under $\kappa - \mu$ extreme fading channels, such as energy detection in terms of area under the receiver operating characteristic curve and outage probability in interference limited scenarios.

Introduction: The $\kappa - \mu$ extreme distribution was originally introduced in [1] as a particular case of the $\kappa - \mu$ distribution. The $\kappa - \mu$ distribution is a generalised fading distribution model which is defined by the fading parameters $\kappa$ and $\mu$. As $\kappa$ increases infinitely and $\mu$ converges to 0, the $\kappa - \mu$ distribution characterises an extreme fading condition which is called the $\kappa - \mu$ extreme distribution. The $\kappa - \mu$ extreme distribution is usually used to characterise severe fading conditions such as a parking lot, a gymnasium and enclosed environments [2]. Despite the usefulness of this distribution, however, researches and channel analysis have been done in only a few studies. In [2, 3], outage probability, bit error rate and mean output signal-to-noise ratio (SNR) have been analysed over the $\kappa - \mu$ extreme fading. In [4], energy detection, especially the average detection probability and receiver operating characteristic (ROC) curve, was investigated over the $\kappa - \mu$ extreme fading.

Hence, we newly formulate closed-form expressions of the generalised moment generating function (G-MGF), which is one of the most essential alternative specifications of distribution probability. The G-MGF can be directly utilised in various application scenarios including energy detection, outage probability, physical layer security, average error rate analysis and so on [5, 6]. To the best of the authors’ knowledge, this is the first derivation of the exact closed-form formula for the G-MGF in the $\kappa - \mu$ extreme fading channels. Additionally, the corresponding performance analysis of several applications, related to energy detection and outage probability, is also presented.

G-MGF of the $\kappa - \mu$ extreme distribution: The G-MGF of a random variable $X$ is defined as [5]

$$\phi_X^{(n)}(s) = \mathbb{E}[X^n e^{sX}] = \int_0^\infty x^n e^{sX} f_X(x) \, dx,$$

where $f_X(x)$ is the probability density function (PDF) of $X$. The PDF of the instantaneous SNR over $\kappa - \mu$ extreme distribution is described as [2]

$$f_X(y) = \frac{2m}{\sqrt{2\pi}} e^{-\frac{2m(\gamma-\gamma_0)}{\sqrt{2}}} I_1\left(\frac{4m}{\sqrt{2}} \sqrt{\frac{\gamma_0}{\gamma}}\right) + \frac{e^{-2m}}{\sqrt{2\pi}} \delta(y),$$

where $m$ denotes fading severity, $I_1(\cdot)$ is modified bessel function [7, Eq. (8.406.1)], and $\delta(\cdot)$ is the Dirac delta function. Now, we can formulate the G-MGF of the $\kappa - \mu$ extreme distribution as below

$$\phi_X^{(n)}(s) = \int_0^\infty y^n e^{sX} f_X(y) \, dy \tag{3}$$

$$= \int_0^\infty y^n e^{\frac{2m}{\sqrt{2\pi}} e^{-\frac{2m(\gamma-\gamma_0)}{\sqrt{2}}} I_1\left(\frac{4m}{\sqrt{2}} \sqrt{\frac{\gamma_0}{\gamma}}\right) + \frac{e^{-2m}}{\sqrt{2\pi}} \delta(y)} \, dy. \tag{4}$$

For $n \geq 1$, the Dirac delta function can be ignored and the latter integral can be reformulated with the aid of [7, Eq. (6.643.2)]. Then, the alternative expression can be also described by using [7, Eq. (9.220.2)] as

$$\phi_X^{(n)}(s) = \frac{2m e^{-2m}}{\sqrt{2\pi}} \int_0^\infty y^n e^{\frac{2m}{\sqrt{2\pi}} e^{-\frac{2m(\gamma-\gamma_0)}{\sqrt{2}}} I_1\left(\frac{4m}{\sqrt{2}} \sqrt{\frac{\gamma_0}{\gamma}}\right) + \frac{e^{-2m}}{\sqrt{2\pi}} \delta(y)} \, dy \tag{5}$$

$$= e^{-2m} n! e^{-\frac{m^2}{2}} \frac{\gamma}{2m-s\gamma} \mathcal{M}_{-n,-1} \left(\frac{4m^2}{2m-s\gamma}\right), \tag{6}$$

where $\mathcal{M}_{-n,-1}(\cdot)$ denotes the Whittaker function [8, Eq. (9.220.2)] and the Kummer confluent hypergeometric function [7, Eq. (13.1.2)], respectively. For $n = 0$, however, we have to formulate the G-MGF with consideration of the Dirac delta function, i.e. the non-nil probability, as

$$\phi_X^{(0)}(s) = e^{-2m} n! e^{-\frac{m^2}{2}} \frac{\gamma}{2m-s\gamma} \mathcal{M}_{-n,-1} \left(\frac{4m^2}{2m-s\gamma}\right) + e^{-2m} \tag{8}$$

$$= e^{-2m} n! e^{-\frac{m^2}{2}} \frac{\gamma}{2m-s\gamma} \mathcal{M}_{-n,-1} \left(\frac{4m^2}{2m-s\gamma}\right). \tag{9}$$

Performance analysis for energy detection: Energy detection is the sensing method of distinguishing the presence or absence of an unknown signal, by comparing the energy of the received signal with a predefined threshold $\lambda$ [4]. To analyse the performance of energy detection, ROC and area under the ROC curve (AUC) [9] are mainly used as metrics, which are defined by the detection probability $P_d$ and the false alarm probability $P_f$. The metrics $P_d$ and $P_f$ can be described as [9]

$$P_d(\gamma, \lambda) = Q_d(\sqrt{2\gamma}\sqrt{\lambda}), \tag{10}$$

$$P_f(\lambda) = \frac{\Gamma(u, \lambda/2)}{\Gamma(u)}, \tag{11}$$

where $Q_d(\cdot, \cdot)$ is the generalised Marcum $Q$-function [10, Eq. (13.1.2)] and $u$ can be obtained through multiplying the one-sided bandwidth by the observing duration. Since the ROC curve depicts $P_d$ according to $P_f$, we need $P_d$ which can be obtained with the aid of [5, Eq. (14)] as

$$P_d = \sum_{n=0}^\infty \frac{\Gamma(u+n, \lambda/2)}{\Gamma(u+n)} \left|\phi_X^{(n)}(s)\right|_{s=1}.$$ 

However, ROC curve is not suitable for comparing two energy detectors whose performance curves are crossing each other and not able to encapsulate $P_d$ and $P_f$ at once. Hence, the AUC was proposed as an alternative metric [9]. The AUC is calculated by integrating the area under the ROC curve as

$$\mathcal{T} = \int_0^1 P_d(\gamma, \lambda) dP_f(\lambda). \tag{13}$$

The AUC can be also easily formulated by G-MGF as [5, Eq. (22)]

$$\mathcal{T} = 1 - \sum_{q=0}^{n-1} \frac{\sum_{u=0}^q} \frac{q - u - 1} \frac{q - n} \left(\frac{1}{2}\right) \frac{\Gamma(u+n, \lambda/2)}{\Gamma(u+n)} \left|\phi_X^{(n)}(s)\right|_{s=1}.$$ 

Outage probability for maximal ratio combining (MRC) systems with interference: In this section, we evaluate the outage probability of $L$-branch using MRC with $N$ interferences. We assume that the channel of the desired signal is affected by the $\kappa - \mu$ extreme distribution and the interference channels undergo Rayleigh fading. We consider the outage as the case that the signal-to-interference ratio (SIR) goes below a given threshold $\gamma_0$. Then, the outage probability can be formulated as

$$P_{out} = Pr(\Gamma < \gamma_0) = 1 - \int_0^\infty F_X(x) f_\Theta(x) \, dx,$$

where $\Gamma$ is SIR, $F_X(x)$ and $f_\Theta(x)$ are the cumulative density functions of the output interfering signal and the PDF of the desired signal, respectively. Then, the outage probability is evaluated by the
G-MGF as [5, Eq. (28)]

\[ P_{\text{out}} = \sum_{l=1}^{N} P_{l} \sum_{m=1}^{2} \frac{1}{(2m)!} \left( \sum_{n=1}^{L} \frac{L}{n_{n}} q_{m} \right) \]

where \( J \) denotes the number of different interfering signals with power \( P_{l}, l = 1, \ldots, N \), \( n_{l} \) is multiplicity of \( P_{l} \), i.e., \( \sum_{n=1}^{J} n_{n} = N \), \( \gamma_{m} \) describes the power of the desired signal at \( m \)th receive antenna, \( \tau(l, L) \) and \( A_{\theta} \) are defined as [5]

\[ \tau(l, L) = \left\{ (q_{1}, q_{2}, \ldots, q_{L}) : q_{n} \in \mathbb{N}, \sum_{n=1}^{L} q_{n} = l \right\} \]

\[ A_{\theta} = \left( -1 \right)^{l-1} \sum_{\Omega_{l}} \frac{L}{l_{n}} \sum_{i=1}^{J} (n_{i} + q_{i} - 1) \]

\[ \times \frac{p_{l} p_{n}}{(P_{l} - P_{n})^{l+n+m}}, \]

where \( \Omega_{l} \) is the set \( \{ (q_{1}, q_{2}, \ldots, q_{L}) : q_{n} \in \mathbb{N}, q_{i} = 0, \sum_{i=1}^{L} q_{i} = l - 1 \} \) and \( \mathbb{N} \) denotes the set of non-negative integers.

**Numerical results:** In this section, the applications based on the G-MGF, the average AUC and the outage probability, are evaluated over the \( \kappa - \mu \) extreme fading in various conditions.

Fig. 1 shows the average AUC with respect to \( \gamma \) when \( u = 1, 5 \). As can be seen in the Fig. 1, the average AUC increases as \( m \) increases. We can also check that the AUC performance is degraded for higher \( u \).

In Fig. 2, we analyse the outage probability with interference under MRC systems. We assume that the number of receive antennas \( L = 2, 4 \), the number of interfering signal \( N = 4 \) and the number of different interferences \( J = 2 \) with \( n_{1} = 2, n_{2} = 2 \). The corresponding interfering power \( P_{1}, P_{2} \) are 0.1, 0.4, respectively, which are normalised by the threshold \( \gamma_{m} \). Trivially, we can observe that as the number of receive antennas increases, the outage probability decreases. Moreover, Fig. 2 also shows lower outage probability with higher \( m \) parameter.

**Conclusions:** In this Letter, we have derived novel closed-form formulas of the G-MGF for the \( \kappa - \mu \) extreme distribution. By exploiting the derived formulas, we have analysed energy detection scenarios with AUC curve and evaluated the outage probability of the system using MRC with inferences.

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